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COMMENT

Anisotropic bond percolation for some 2D lattices

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Abstract. The anisotropic bond percolation on the triangular, square and honeycomb lattices is studied uniformly. We use a longitudinal cell to discuss the triangular lattice and then from this we obtain the square and honeycomb lattices. The fixed points, the critical surfaces, the flow diagrams and the correlation exponents are given. The results are satisfactory.

An anisotropic bond percolation is one of the extensions of the usual bond percolation model. Many authors have studied an anisotropic bond percolation on a square lattice by RSRG (see e.g. Chaves *et al* 1979, Nakanishi *et al* 1981, Oliveira 1982). They obtained the fixed points, flow diagrams and the critical exponents of both the correlation length and the dimensionality crossover.

Guttmann and Whittington (1982) have proposed bond–site percolation on an anisotropic triangular lattice. They divided the bonds on the triangular lattice into two classes with the different occupation probabilities p, d and allowed the sites to be occupied independently with the probability s. The critical surface and the exponents ν have been obtained.

In contrast to the above, the pure bond percolation on an anisotropic triangular or honeycomb lattice has been studied far less. In this comment we discuss anisotropic bond percolation for some two-dimensional lattices. Firstly we study a triangular lattice then by a duality transformation we obtain the honeycomb problem. As a special case when the occupation probability of one class of bonds on the triangular lattice equals zero, the square lattice is obtained. Finally we have a chain limitation from the honeycomb lattice. For all these lattices, we obtain the fixed points, the critical surfaces or flow figures and the correlation length exponents.

We consider a triangular lattice and divide all bonds into three classes on the different spatial orientations. These bonds are occupied independently with the probabilities p, q and r (figure 1). We choose a longitudinal cluster as an elementary cell ($b=1$) which consists of three bonds with different occupation probabilities and has the shape of a parallelogram. The $b=2$ renormalised cell will have twelve bonds and the same shape as the elementary cell (figure 2). By using the deletion–contraction rule to evaluate the equivalent probability (de Magalhães *et al* 1981, Tsallis and Levy 1981), we obtain the equation

$$\begin{aligned} p' = {} & pq + 2pr + p^2q + p^2r + 2pqr^2 + pq^2r - 5p^2qr - p^2q^2 \\ & -2p^2r^2 + p^3q - 2pqr^3 - 4pq^2r^2 - 3p^2qr^2 - 3p^3qr \\ & -p^3r^2 + 2pq^2r^3 + 4p^2qr^3 + 4p^2q^2r^2 - p^2q^3r + 9p^3qr^2 \end{aligned}$$

$$
\begin{aligned}
&+4p^3q^2r+p^3r^3-p^4qr-2p^4q^2-p^2q^2r^3+4p^2q^3r\\
&-4p^3qr^3-2p^3q^2r^2+p^3q^3r+p^4qr^2+3p^4q^2r+p^4q^3\\
&-2p^2q^3r^3-2p^3q^2r^3-7p^3q^3r^2-p^4qr^3-7p^4q^2r^2\\
&-3p^4q^3r+p^5q^2r+3p^3q^3r^3+4p^4q^2r^3+7p^4q^3r^2\\
&-2p^4q^3r^3-p^5q^3r^2\\
\equiv{}& f(p,q,r). \qquad (1)
\end{aligned}
$$

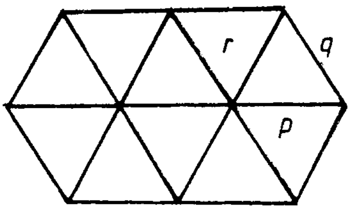

Figure 1. A triangular lattice with the different occupied probabilities p, q and r on the different spacial orientations of bonds.

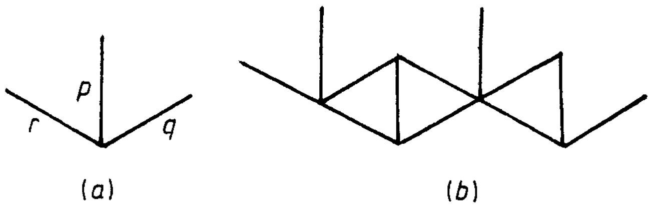

Figure 2. (*a*) The $b=1$ cell consists of three bonds with the occupied probabilities p, q and r. (*b*) The renormalised cell ($b=2$) consists of twelve bonds.

Then we choose the same cells on the other two directions of the triangular lattice. The equations are

$$q'=f(q,r,p) \qquad (2)$$

$$r'=f(r,p,q). \qquad (3)$$

These are the renormalisation group equations. The fixed points given by the RG transformation are $(1,1,1)$, $(0,0,0)$, $(0,1,1)$, $(1,0,1)$, $(1,1,0)$ and $(0.3368, 0.3368, 0.3368)$. The first two points are trivial and the next three are, in practice, the fixed points on the square lattice. The final fixed point (I) is an isotropic one on the anisotropic triangular lattice.

The critical surface for the (p, q, r) parameter space is shown in figure 3. It is obtained numerically by following flows from some points. This surface divides the parameter space into the percolating region and the unpercolating one. Flows from points in the percolating region are into $(1, 1, 1)$ and from points in the unpercolating into $(0,0,0)$. The shape the of critical surface is in agreement with the result from the exact critical condition but there are some numerical differences.

The fixed point (I) is unstable on the diagonal which is from $(0,0,0)$ to $(1,1,1)$ and it is stable along the critical surface. This result is analogous with the square lattice.

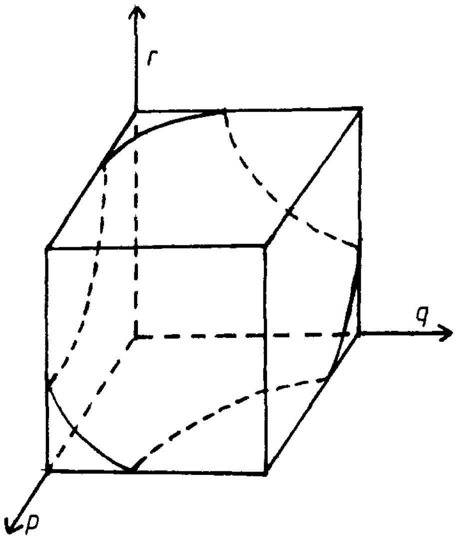

Figure 3. The critical surface for the triangular lattice.

The linear transformation matrix about (I) is

$$T = (\partial(p', q', r')/\partial(p, q, r))_{(I)}$$

with the maximum eigenvalue $\lambda_1 = 1.7410$. The correlation length exponent is given by

$$\nu = \ln b/\ln \lambda_1 = 1.25.$$

These results agree with the exact values $p_c = 0.3473$ and $\nu = 1.34$ to within 3% and 7% error.

To obtain percolation on the anisotropic square lattice, we let $q = 0$ in the equation (1) and obtain

$$p' = f(p, 0, r) \equiv g(p, r). \tag{4}$$

The other equation given by the exchange $p \leftrightarrow r$ is

$$r' = g(r, p). \tag{5}$$

These recursion relations give the trivial (stable) fixed points (1,1), (0, 0) and the isotropic fixed point (0.5249, 0.5249). The exponent $\nu = 1.42$. The flow diagram of equations (4) and (5) is given in figure 4.

We note that we can not let $q = 0$ in equations (1) and (3) simultaneously to obtain the recursion relations for the square lattice. This is because when $q = 0$ in both (1) and (3) they represent a different shape of the cells to the square lattice. Clearly this is not correct.

Next we make the duality transformation on equations (1), (2) and (3). We shall get the renormalisation group equations for the honeycomb lattice. (To enable the effect of the duality transformation to be easily understood we use the present form of the

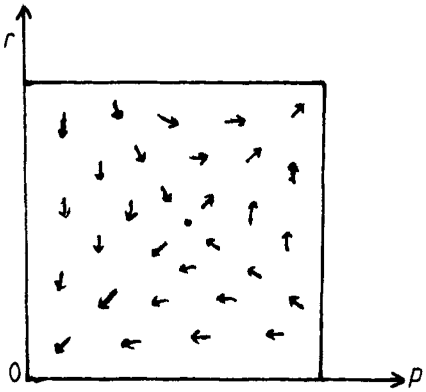

Figure 4. The flow diagram for the square lattice.

RG equations.)

$$
\begin{aligned}
p' = 1 &- (1-p)^2 - p(1-p)(1-r)(q+r-qr) - p(1-p)(p+r-pr)(q+r-qr) \\
&- p(1-p)(1+p)(1-q)(1-r) - p(1-p)^2(1-q) \\
&+ pqr(1-p)(q+r-qr)^2(p+r-pr) \\
&+ pqr^2(1-p)(1+p)(1-q)(1-r)(q+r-qr) \\
&+ qr^2(1-p)^2(q+r-qr) - qr^2(1-p)^3(1-qr)(q+r-qr) \\
\equiv\ & h(p, q, r)
\end{aligned}
\tag{6}
$$

$$q' = h(q, r, p) \tag{7}$$

$$r' = h(r, p, q). \tag{8}$$

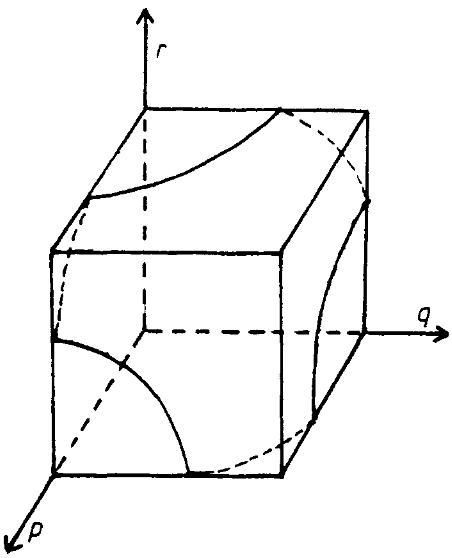

Figure 5. The critical surface for the honeycomb lattice.

The fixed points are $(p^*, q^*, r^*) = (0, 0, 0)$, $(1, 1, 1)$, $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$ and $(0.6632, 0.6632, 0.6632)$. The exponent $\nu = 1.27$. The critical surface is shown in figure 5. The physical feature of figure 5 is analogous with the triangular lattice.

When the occupied probability of one class of bonds is unity and the other class is zero on the honeycomb lattice we will reach the chain limitation. Therefore let $q = 1$, $r = 0$ in equation (6), we obtain $p' = p^2$ and $p^* = 1$, $\nu = 1$.

In summary, we have discussed the anisotropic bond percolation for some two-dimensional lattices uniformly. On the whole, the level of agreement of the obtained results with the exact or previous results is satisfactory.

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